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Statistical entropy of de Sitter space

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Abstract: Quantum gravity in 2+1 dimensions with a positive cosmological constant can be represented as an SL(2,C) Chern-Simons gauge theory. The symmetric vacuum of this theory is a degenerate configuration for which the gauge fields and spacetime metric vanish, while de Sitter space corresponds to a highly excited thermal state. Carlip's approach to black hole entropy can be adapted in this context to determine the statistical entropy of de Sitter space. We find that it equals one-quarter the area of the de Sitter horizon, in agreement with the semiclassical formula.

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An inertial observer in the 2+1-dimensional de Sitter vacuum detects thermal radiation at the temperature [1]

$$T = \frac{1}{2\pi\ell}\,. \tag{1}$$

The boundary of the past light cone of the observer's world line is the de Sitter horizon. The horizon area is $\frac{2\pi}{\sqrt{\Lambda}}$, where Λ is the cosmological constant. The corresponding thermodynamic entropy is

$$S = \frac{\pi}{2\sqrt{\Lambda}G}\,, \tag{2}$$

where G is Newton's constant. The microscopic origin of (1) and (2) is an outstanding mystery.

In this paper we will shed some light on this mystery in the context of pure 2+1 gravity [2] . Our approach will be a direct adaptation to the de Sitter case of Carlip's analysis [3] of the 2+1 black hole. 2+1 gravity with a positive cosmological constant is equivalent to an SL(2,C) Chern-Simons gauge theory [4, 5]. We consider this theory on a spatial disc whose boundary corresponds to the de Sitter horizon. After imposition of the horizon boundary conditions proposed in [3], the 2+1 SL(2,C) gauge theory reduces to an SL(2,C) WZW theory on the horizon boundary of the disc [5, 6, 7]. The ground state of this WZW theory has vanishing metric and gauge fields, and corresponds to the elusive state of unbroken symmetry in 2+1 gravity. De Sitter space arises as an excited thermal state in which the metric acquires an expectation value. In this paper we show that the statistical entropy of this state agrees with the thermodynamic entropy (2).

Carlip's derivation rests on several key assumptions. One is the existence of a suitably defined conformal field theory with the SL(2,R)$\times$SL(2,R) (which in our case of positive Λ becomes SL(2,C))[1] current algebra with the required properties. In particular the asymptotic growth of states should depend on the central charge in the familiar manner [9] dictated by unitarity, despite the fact that it is not a unitary theory. A second concerns the nature of the boundary conditions. The choice in [3] does not appear to be unique, and other choices do not yield the desired value for the entropy. In this paper we shall not question any of these assumptions. We will find that in the positive Λ context they do imply the de Sitter entropy (2). The validity of the assumptions certainly warrants further investigation.

2+1 gravity with a positive cosmological constant is described by the action

$$I_{grav} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R - \frac{2}{\ell^2}\right), \tag{3}$$

where the cosmological constant is $\Lambda = \frac{1}{\ell^2}$ and we have omitted surface terms. In the following we will be interested in a semiclassical limit, which requires that the cosmological constant is small in Planck units, or equivalently

$$\ell \gg G\,. \tag{4}$$

[1] The SL(2,C) theory might be defined as the complexification of an SU(2) theory. The SL(2,C) Chern-Simons theory was extensively studied as such in [8], but the status of the theory on the disc remains unclear.

(3) has the de Sitter solution

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\phi^2 \,, \tag{5}$$

where ϕ has period 2π and $0 \leq r \leq \ell$. Spacelike surfaces of constant time t are discs D_2. The coordinate system (5) covers the portion of de Sitter space corresponding to the past light cone of an observer following the geodesic $r = 0$. The boundary of the region of de Sitter space from which no causal signal can reach such an observer is the horizon at $r = \ell$. We seek a microscopic derivation of (2) in the context of the quantum theory of 2+1 gravity on $D_2 \times R$.

The quantization of (3) is best achieved by recasting the theory as an SL(2,C) Chern-Simons gauge theory [4, 5]. The action for this this theory is, up to boundary terms,

$$I_{grav}[A] = \frac{is}{4\pi} \int_{D_2 \times R} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) - \frac{is}{4\pi} \int_{D_2 \times R} Tr(\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A}) \,, \tag{6}$$

where, in the conventions of [3][2]

$$s = \frac{\sqrt{2}\ell}{8G} \,, \tag{7}$$

and A is an SL(2,C) gauge field. (6) is equivalent to (3) with the identification

$$A^a = \omega^a + \frac{i}{\ell} e^a \,, \tag{8}$$

for $a = 0, 1, 2$. Here $e^a = e^a_\mu dx^\mu$ is the triad and $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} dx^\mu$ is the SL(2,R) spin connection. It is easily checked that the de Sitter solution (5) corresponds to

$$\begin{aligned} A^0 &= \sqrt{\frac{1}{2} - \frac{r^2}{2\ell^2}} (-d\phi + \frac{i}{\ell} dt) \,, \\ A^1 &= \frac{i}{\sqrt{2\ell^2 - 2r^2}} dr \,, \\ A^2 &= \frac{r}{\sqrt{2}\ell^2} dt + \frac{ir}{\sqrt{2}\ell} d\phi \,. \end{aligned} \tag{9}$$

The Chern-Simons theory (6) has no local dynamics. The quantum theory on $D_2 \times R$ is defined by imposing boundary conditions on the cylindrical (ϕ, t) boundary at $r = \ell$ which set half of the 6 complex tangential components of the gauge field to fixed values A^B [5, 6, 7]. We wish to choose boundary conditions appropriate to a horizon of proper length $2\pi\ell$. Carlip [3] has argued that the appropriate boundary conditions are[3]

$$\begin{aligned} A^{B0}_\phi + A^{B1}_\phi &= 0 \,, \\ A^{B0}_t + A^{B1}_t &= 0 \,, \\ A^{B2}_\phi &= \frac{\omega}{\sqrt{2}} + \frac{i}{\sqrt{2}} \,. \end{aligned} \tag{10}$$

[2] The gauge group is a complexification of SL(2,R) with generators and structure constants normalized as $TrT^aT^b = 2\eta^{ab}$ and the line element is $ds^2 = 2\eta_{ab} e^a e^b$.

[3] An alternate and simplified version of the boundary conditions can be found in [10] .

This insures that the induced metric on the horizon agrees with (5) and in particular is of proper length $2\pi\ell$. ω here is an arbitrary real constant. All constant values of ω correspond to a horizon on the boundary in some gauge [3]. In the following we will find the value of ω which gives the dominant contribution to the entropy.

Imposing (10) reduces (6) to a complex level $k = is$ SL(2,C) WZW conformal field theory on the boundary of the disc [5, 6, 7, 3, 10], after redefining $A \to \tilde{A}$ by a certain gauge transformation [3]. The currents are defined by the expansion of $\tilde{A}_\phi$ on the boundary

$$\tilde{A}_\phi = \frac{i}{s} \sum j_n e^{in\phi} \,. \tag{11}$$

The SL(2,C) current algebra is

$$\begin{aligned} [j^a_m, j^b_n] &= i\epsilon^{ab}{}_c j^c_{m+n} + ism\frac{1}{2}\eta^{ab}\delta_{m+n,0} \,, \\ [j^a_m, \bar{j}^b_n] &= 0 \,, \\ [\bar{j}^a_m, \bar{j}^b_n] &= i\epsilon^{ab}{}_c \bar{j}^c_{m+n} - ism\frac{1}{2}\eta^{ab}\delta_{m+n,0} \,. \end{aligned} \tag{12}$$

The Virasoro generator L_0 is then given by the Sugawara construction as

$$L_0 = \frac{1}{2is-1} Tr \sum_{n=-\infty}^{n=\infty} : j_n j_{-n} : - \frac{1}{2is+1} Tr \sum_{n=-\infty}^{n=\infty} : \bar{j}_n \bar{j}_{-n} : \,. \tag{13}$$

Writing the nonzero mode piece as level number N this is

$$\begin{aligned} L_0 &= N + \frac{1}{2is-1} Tr j_0^2 - \frac{1}{2is+1} Tr \bar{j}_0^2 \\ &= N - \frac{1}{1+4s^2} Tr(j_0^2 + \bar{j}_0^2) - \frac{2is}{1+4s^2} Tr(j_0^2 - \bar{j}_0^2) \,. \end{aligned} \tag{14}$$

The squares of the zero modes of j are determined by the boundary conditions (10) and their relation (11) to $\tilde{A}$:[4]

$$Tr j_0^2 = -s^2(\omega + i)^2 \,. \tag{15}$$

Inserting this relation yields

$$L_0 = N + \frac{2s^2(\omega - 2s)^2}{1+4s^2} - 2s^2 \,. \tag{16}$$

The Wheeler-de Witt equation implies $L_0 = 0$ [3]. The maximal value of N is then attained for $\omega = 2s$ as

$$N = 2s^2 = \frac{\ell^2}{16G^2} \,. \tag{17}$$

In the semiclassical regime of large s the central charge may be approximated by $c \sim 6$. The unitarity relation for the logarithm of the number of states satisfying (17) is to leading order in s

$$S = 2\pi\sqrt{\frac{cN}{6}} = \frac{\pi\ell}{2G} \,, \tag{18}$$

[4] The $A \to \tilde{A}$ redefinition does not affect this relation.

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in agreement with the semiclassical formula (2).

A computation of the de Sitter entropy in a fully dynamical theory such as string theory remains an outstanding challenge.

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